MATH 2020 Advanced Calculus II

Tutorial 2

1. Compute $\displaystyle\int_1^{e^3}\int_{\ln y}^{3} e^{e^x-x}\,dxdy$.

   Solution. Since we do not know how to integrate $\int e^{e^x-x}dx$, we try to reverse the order of integration:

$$\begin{aligned}
\int_1^{e^3}\int_{\ln y}^{3} e^{e^x-x}\,dxdy &= \int_0^3\int_1^{e^x} e^{e^x-x}\,dydx\\
&= \int_0^3 \big[y\big]_1^{e^x} e^{e^x-x}\,dx\\
&= \int_0^3 (e^x-1)e^{e^x-x}\,dx\\
&= \int_0^3 e^{e^x-x}\,d(e^x-x)\\
&= e^{e^3-3}-e.
\end{aligned}$$

2. Compute $\displaystyle\int_0^1\int_{x^3}^{1} \frac{x\cos(\sqrt[6]{y})}{\sqrt{y}^3}\,dydx$.

   Solution. Since we do not know how to integrate $\int \frac{\cos(\sqrt[6]{y})}{\sqrt{y}^3}dy$, we try to reverse the order of integration:

$$\begin{aligned}
\int_0^1\int_{x^3}^{1} \frac{x\cos(\sqrt[6]{y})}{\sqrt{y}^3}\,dydx &= \int_0^1\int_0^{\sqrt[3]{y}} \frac{x\cos(\sqrt[6]{y})}{\sqrt{y}^3}\,dxdy\\
&= \int_0^1 \left[\frac{x^2}{2}\right]_0^{\sqrt[3]{y}} \frac{\cos(\sqrt[6]{y})}{\sqrt{y}^3}\,dy\\
&= \frac{1}{2}\int_0^1 y^{\frac{2}{3}-\frac{3}{2}}\cos(\sqrt[6]{y})\,dy\\
&= \frac{1}{2}\int_0^1 y^{-\frac{5}{6}}\cos(\sqrt[6]{y})\,dy\\
&= \frac{1}{2}\times\left[6\sin(\sqrt[6]{y})\right]_0^1\\
&= 3\sin 1.
\end{aligned}$$

3. Find the volume of the solid in the first octant of $\mathbb{R}^3$ bounded by $x^2 + y^2 = 1$ and $x + z = 1$.

Solution. The volume is equal to

$$\begin{aligned}
&\int_0^1 \int_0^{\sqrt{1-x^2}} (1-x)dydx \\
&= \int_0^1 \sqrt{1-x^2}(1-x)dx \\
&= \int_0^1 \sqrt{1-x^2}dx - \int_0^1 x\sqrt{1-x^2}dx \\
&= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta d\theta + \left[\frac{1}{3}\sqrt{1-x^2}^3\right]_0^1 \qquad \text{(Sub. } x = \sin\theta \text{ for the 1st integral)} \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)d\theta - \frac{1}{3} \\
&= \frac{\pi}{4} - \frac{1}{3}.
\end{aligned}$$

4. Compute $\int_0^2 \int_0^{\frac{y}{2}} (x^2+y^2)dxdy + \int_2^3 \int_0^{3-y} (x^2+y^2)dxdy$ by reversing the order of integration.

Solution. Notice that we are integrating over the region bounded by $x = 0,\ y = 2x$ and $x + y = 3$. By reversing the order, the integral becomes

$$\begin{aligned}
&\int_0^1 \int_{2x}^{3-x} (x^2+y^2)dydx \\
&= \int_0^1 x^2(3-x-2x) + \frac{1}{3}\left[(3-x)^3 - (2x)^3\right]dx \\
&= \left[x^3 - \frac{3}{4}x^4 - \frac{1}{12}(3-x)^4 - \frac{8}{3}\cdot\frac{x^4}{4}\right]_0^1 \\
&= 1 - \frac{3}{4} - \frac{16-81}{12} - \frac{2}{3} \\
&= 5.
\end{aligned}$$